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Knuth's (1, 2, 1) Unstacking

This presentation is dedicated to Donald Knuth who has proposed many interesting and challenging problems in the Problems Section of *The American Mathematical Monthly*. The problem considered is that proposed by Barry Hayes, Knuth, and Carlos Subi (**E3267 [1988,456]**). The published solution, due to Albert Nijenhuis, just recently appeared in the March 1993 *Monthly*, pages 292-294.

The problem is as follows. Suppose that we are given n piles of blocks; the i-th pile having a_i blocks, $i = 1, 2, \ldots, n$. Dismantle the piles by choosing a pile having 2 or more blocks, removing 2 blocks and putting one block on the pile on the left and one block on the pile on the right. Continue the process until there are no more piles having more than one block. What is the final configuration like and how many moves are required?

My solution differs from that of Nijenhuis in that it pays closer attention to the evolution of the piles as the dismantling progresses. I also generalize to the $(s, 2s, s)$ unstacking, where $s \geq 1$.

Problem E3267

Given a sequence $(x_1, x_2, \ldots, x_l)$ of nonnegative integers in which $x_k > 1$ for some k, let us say that a "k-move" is the operation of replacing the subsequence (x_{k-1}, x_k, x_{k+1}) by $(x_{k-1} + 1, x_k - 2, x_{k+1} + 1)$.

(a) Prove that repeated application of such moves to the sequence $(0^m, 2m, 0^m)$ always leads to the sequence $(1^m, 0, 1^m)$ after exactly $\frac{1}{3}(m+1)(m+\frac{1}{2})m$ moves. Here 0^m and 1^m stand for sequences of m 0's and m 1's, respectively.

(b) Prove that, for sufficiently large m, the starting sequence $(0^m, a_1, a_2, \ldots, a_n, 0^m)$ leads inexorably to the sequence $(0^{m+p}, 1^q, 0, 1^r, 0^{m+n-p-q-r-1})$ for some p, q, and r, if $a_1, a_2, \ldots, a_n$ are positive integers. Furthermore, p, q, and r can be expressed in terms of $\sum_{j=1}^n j$ and $\sum_{j=1}^n j a_j$.How many moves does this transformation require?

Solution:

For a given sequence $\mathrm{x} = (x_1, x_2, \ldots, x_l)$ of nonnegative integers, define $K(\mathrm{x}) = \sum_{j=1}^l x_j$, and $L(\mathrm{x}) = \sum_{j=1}^l j x_j$, and finally $M(\mathrm{x}) = \sum_{j=1}^l j^2 x_j$. For the fixed $\mathrm{a} = (a_1, a_2, \ldots, a_n)$, we shall use the letters G, H, J to denote $K(\mathrm{a}), L(\mathrm{a})$, and $M(\mathrm{a})$ respectively. Agree that 1^0 means the empty word, $\emptyset$.

In order to get insight into the problem, consider the following more easily described "task". Supposed that we are given n piles of blocks having $a_1, a_2, \ldots, a_n$ blocks respectively. Suppose that we dismantle these piles by using elementary moves described as follows: if a given pile has more than one block in it, remove a block and place it on the pile(possibly empty) immediately on its right. Put more formally, given a sequence $(x_1, x_2, \ldots, x_l)$ and a number $0 < k < l$, a right k-move is the operation of replacing the pair (x_k, x_{k+1}) by $(x_k - 1, x_{k+1} + 1)$. Intuitively, we can easily imagine what will happen if we repeatedly perform right k-moves on the n piles represented by a. We inexorably arrive at a sequence of G piles each consisting of exactly one block.

To probe a little further, suppose that we regularize the dismantling process by doing a scanning process from the right which consists of performing several right k-moves in succession. If the piles are in some state, scan the piles from the right until we reach a pile having more than one block. Perform a right k-move and then proceed to the right to the next pile and perform another right k-move, if possible, until one reaches an empty pile. As the process proceeds, we form "stepping stones" of one block each over which we transfer single blocks to the end of the line. Put more mathematically, if $\mathrm{x} = wb1^\alpha$, where w is a word of integers and $b > 1$, then $scan(\mathrm{x}) = w(b-1)1^{\alpha+1}$. Notice that $scan^b(\mathrm{x}) = w1^{\alpha+b}$. We continue the process by operating on a shorter word than w. If we let $\mathrm{x}_0 = (a_1, a_2, \ldots, a_n 0^{G-n})$ to represent the original piles, then $\mathrm{x}_1 = (1^G)$ represents the terminal state.

It is also easy to calulate the number of moves required because of the following facts. If x is a sequence of integers and f_k is a right k-move then

$$K(f_k(\mathrm{x})) = K(\mathrm{x})$$
$$L(f_k(\mathrm{x})) = L(\mathrm{x}) + 1.$$

Thus, to calculate the number of right k-moves necessary we merely calculate $L(\mathbf{x}_1) - L(\mathbf{x}_0) = \sum_{j=1}^{G} j - H = G(G+1)/2 - H$. We could also dismantle the piles by left k-moves, but then $\mathbf{x}_0 = (0^{G-n}, a_1, a_2, \ldots, a_n)$ and $\mathbf{x}_1 = (1^G)$, while the number of moves is the same.

For the problem posed, the situation is more interesting. Again we can consider the task of dismantling the given piles, but the k-moves are symmetric this time. Also, notice that if $\mathbf{x} = (x_1, x_2, \ldots, x_l)$ and f_k is a k-move, then

$$\begin{aligned} K(f_k(\mathbf{x})) &= K(\mathbf{x}) \\ L(f_k(\mathbf{x})) &= L(\mathbf{x}) \\ M(f_k(\mathbf{x})) &= M(\mathbf{x}) + 2. \end{aligned}$$

The first step in the solution of the problem is to choose the number, m, of blank piles to put on each side of the given piles. Let us be conservative at the beginning and choose $m = G$. The tuple describing the initial state is thus

$$\mathbf{x}_0 = (0^G, a_1, a_2, \ldots, a_n, \quad 0^G).$$

The notation $\mathbf{x} \rightarrow \mathbf{y}$ means that $\mathbf{y}$ is obtained from $\mathbf{x}$ by some successive k-moves. Let

$$\mathbf{x}_1 = (0^{m+p}, 1^q, 0, 1^r, 0^{m+n-p-q-r-1})$$

where p, q, r are yet to be chosen. We show that, if $\mathbf{x}_0 \rightarrow \mathbf{x}_1$ then p, q, r are unique.

Lemma 1: If $\mathbf{x} = (0^{s_1}, 1^{q_1}, 0, 1^{r_1}, 0^{t_1})$ and $\mathbf{y} = (0^{s_2}, 1^{q_2}, 0, 1^{r_2}, 0^{t_2})$ are each obtained from $\mathbf{x}_0$ by a sequence of k-moves, then $\mathbf{x} = \mathbf{y}$.

Proof: We know that

$$\begin{aligned} q_1 + r_1 = q_2 + r_2 &= G \\ s_1 + q_1 + r_1 + t_1 = s_2 + q_2 + r_2 + t_2 &= 2G + n. \end{aligned}$$

Now $L(\mathbf{x}) = L(\mathbf{y})$. So

$$\sum_{j=s_1+1}^{s_1+q_1} j + \sum_{j=s_1+q_1+2}^{s_1+q_1+r_1+1} j = \sum_{j=s_2+1}^{s_2+q_2} j + \sum_{j=s_2+q_2+2}^{s_2+q_2+r_2+1} j.$$

But then

$$\frac{(2s_1 + q_1 + r_1 + 2)(q_1 + r_1 + 1)}{2} - (s_1 + q_1 + 1) = \frac{(2s_2 + q_2 + r_2 + 2)(q_2 + r_2 + 1)}{2} - (s_2 + q_2 + 1).$$

Some straight-forward algebraic manipulations and the fact that $q_1+r_1 = q_2+r_2 = G$ yields

$$(s_1 - s_2)G = q_1 - q_2.$$

Since $0 \le q_1, q_2 \le G$, we find that $-G \le (s_1 - s_2) \le G$ or $-1 \le s_1 - s_2 \le 1$. In case, $s_1 = s_2$, we find that $q_1 = q_2$ So, $r_1 = r_2$ and $t_1 = t_2$ and thus $\mathbf{x} = \mathbf{y}$.

If $s_1 - s_2 = 1$, then $q_1 - q_2 = G$ and thus $q_1 = G$ while $q_2 = 0$ and hence $r_1 = 0$ and $r_2 = G$. Also $t_2 - t_1 = 1$. Thus

$$\mathbf{x} = (0^{s_1}, 1^G, 0, \emptyset, 0^{t_1}) = (0^{s_1}, 1^G, 0^{t_1+1}) = (0^{s_2}, \emptyset, 0, 1^G, 0^{t_2}) = \mathbf{y}.$$

The net result of the lemma is that we have reduced the problem to finding some regular ways to dismantle a sequence of piles of blocks. There are three such processes which come readily to mind. Let $\mathbf{x}$ be the vector representing the state of the piles at a given time.

Full Scanning
For i=1 to 2G+n
If $x_i < 2$ then next i
Else do an i-move
Next i
Call the result $scan(\mathbf{x})$.

Right Scanning
Move from right to left until we find an k such that $x_k > 1$.
Now do a scan as described above but starting at k.
Call the next result $Rscan(\mathbf{x})$.

Left Scanning
The process is the same as left scanning only we proceed from the left to the first k such that $x_k > 1$, but then we scan left from k. Call the next result $Lscan(\mathbf{x})$.

We shall use right scanning in our argumentation.

There are four kinds of states $\mathbf{x}$ which will be of interest. Let a, b, c, d be positive integers with $d > 1$ and let w be a word of nonnegative integers.

Type A Here $\mathbf{x} = (w, u, d, 0^a)$.

Type B Here $\mathbf{x} = (w, u, d, 1^b, 0^a)$. $\mathbf{x}_0$ is of type A or B.

Type C Here $\mathbf{x} = (w, u, d, 0, 1^c, 0^a)$.

Type D Here $\mathbf{x} = (w, u, d, 1^b, 0, 1^c, 0^a)$.

Notice the following important changes that take place as we begin with $\mathbf{x}_0$ and repeatedly perform right scans.

If $\mathbf{x}$ is of type A, then $Rscan(\mathbf{x}) = (w, u+1, d-2, 1, 0^{a-1})$ Notice that the result decreases d by 2 and a by 1. If $d = 3$ or $d = 2$ and $u = 0$ and $w = 0^e$ for some e we are finished. If $d = 3$ or $d = 2$ and $u > 0$, then $Rscan(\mathrm{x})$ is of type B or C with a shorter word w_1 than w.If $d > 3$ then $Rscan(\mathrm{x})$ is of type B.

If x is of type B, then $Rscan(\mathrm{x}) = (w, u+1, d-1, 1^{b-1}, 0, 1, 0^{a-1})$ and we accomplish this in $b+1$ k-moves.In physical terms, this means that if we start with some piles of blocks having a string of b single blocks preceeded by a pile of more than 1 block, by performing a sequence of k-moves to the right, we use the b "stepping stones" to transfer a single block to the end of the piles always leaving an empty pile immediately behind us and having a pile of two blocks on the pile immediately before us until we get to the end of the line where we put down the last block in a newly-created space but we leave an empty space immediately preceeding. The net result is that nothing is accomplished in the dismantling process, but a new space is created for a new block a some later stage. Notice that if $u = 0$ and $d = 2$ we are done. Otherwise, $Rscan(\mathrm{x})$ is of type D.

If x is of type C, then $Rscan(\mathrm{x}) = (w, u+1, d-2, 1^{c+1})$. If $d = 2$ or $d = 3$, $Rscan(\mathrm{x})$ is of type C or B respectively but with a shorter word w_1 involved. Of course, this is done in 1 k-move.

If x is of type D, then $Rscan(\mathrm{x}) = (w, u+1, d-1, 1^{b-1}, 0, 1^{c+1}, 0^a)$, again in $b+1$ moves. Again, in physical terms, we can use the b stepping stones in our transfer of the single block but we cannot get to the end this time but merely place the block in the empty space before us, leaving an empty pile behind us. Notice that if $d = 2$, $Rscan(\mathrm{x})$ is of type D. Also, if $d > 2$ and $b = 1$, $Rscan(\mathrm{x})$ is of type C. Otherwise $Rscan(\mathrm{x})$ is of type D is of type D but with smaller b.

There is a nice way to see what is happening as we continue to do successive right scans. Associate with each state x of type A the lattice point $(0,0)$, of type B the lattice point $(b,0)$, of type C the point $(0,c)$ and, finally, of type D the point (c,d). Now x_0 is of type B or C and thus is a point on the x-axis. As we successively scan a state associated with a point on the x-axis we proceed to lattice points on the line $x+y=b$ which are associated with states of type D. When we reach the point $(0,b)$ which is of type C we proceed in one k-move to the point $(0,b+1)$. At each stage we deplete the 'd-pile' by 1 except in our passage from type C to B where we depete the 'd-pile' by 2. Notice that if we reach the stage where $d = 2$ and if $x = (w, u, 2, 0, 1^c, 0^a)$,then $Rscan(\mathrm{x}) = (w, u+1, 1^{c+1}, 0^a)$ and we begin right scanning a word of shorter length. Otherwise we reach a state where $d = 1$ and thus if $\mathrm{x} = (w, u, 2, 1^c, 0, 1^d, 0^a)$, $Rscan(\mathrm{x}) = (w, u+1, 1^{c+1}, 0, 1^d, 0^a)$ which is associated with the lattice point $(c+1, b)$, thus moving horizontally one unit. Again, we begin scanning a state where the word w is of shorter length. The net result is a "Cantor-like walk" through the lattice until the word w is of the form 0^v for some v. Since the function K is preserved under k-moves we must reach a point (q,r) in the lattice such that $q+r=G$ and, of course, the point (q,r) is in the first quadrant.

The preceeding analysis proves the following theorem.

Theorem: Let $\mathrm{x}_0 = (0^G, a_1, a_2, \ldots, a_n, 0^G)$ describe the original state of the piles of blocks. Then there is a positive integer v and nonnegative integers q and r with $q+r=G$ and an integer p with $0 < p < G$ such that

$$Rscan^v(\mathrm{x}_0) = (0^{G-p}, 1^q, 0, 1^r, 0^{G+n+p-q-r-1}).$$

Similarly, there is a positive integer v_1 and nonnegative integers q_1 and r_1 with $q_1 + r_1 = G$ and an integer p_1 with $0 < p_1 \leq G$ such that

$$Lscan(\mathbf{x}) = (0^{G+n+p_1-q_1-r_1-1}, 0^{r_1}, 0, 0^{q_1}, 0^{G-p_1}).$$

We can now use the functions, K, L, M to answer our remaining questions. The fact that a K and L are preserved under k-moves allows us to find the numbers q and r in terms of the number p. The requirement that the numbers q and r satisfy the conditions that $0 < q \leq G$ and $0 \leq r \leq G$ enable us to determine what the number p is. The fact that M is increased by 2 under a k-move enables us to calculate the number of k-moves required to change the initial state $\mathbf{x}_0$ to the final state $\mathbf{x}_1$. We merely have to calculate

$$\frac{L(\mathbf{x}_1) - L(\mathbf{x}_0)}{2}.$$

Theorem: Let

$$\mathbf{x}_0 = (0^G, a_1, a_2, \ldots, a_n, 0^G)$$

be the initial state and let

$$\mathbf{x}_1 = (0^{G-p}, 1^q, 0, 1^r, 0^{G+n+p-q-r-1})$$

the final state after dismantling by k-moves. Then, $q + r = G$ and

$$q = \frac{2G - 2H + G(G+1) - 2pG}{2}$$
$$r = \frac{2H - G(G+1) + 2pG}{2}.$$

Since K is preserved under k-moves, we know that

$$q + r = G. \tag{1}$$

Since L is preserved we also have

$$\sum_{j=G+1}^{G+n} j a_{j-m} = \sum_{j=G-p+1}^{G-p+q} j + \sum_{G-p+q+2}^{G-p+q+1+r} j.$$

Using the formulas for the sum of an arithmetic progression we find that

$$H + G^2 = \frac{q(2G - 2p + q + 1) + r(2G - 2p + 2q + r + 3)}{2}. \tag{2}$$

Using the fact that $q+r = G$, that $q+3r = G+2r$, and also that $(q+r)^2 = G^2 = q^2+2qr+r^2$ we find that

$$H = -pG + r + G(G+1)/2.$$

From this the two equations for q and r follow.

The equations for q and r contain the variable p, but the variable p is not fully determined.To remove the indeterminacy, use the fact that assigning different values to p gives different values for q and r but these values are congruent modulo G. Thus, we merely choose p so that the corresponding values for q and r are such that $0 < q \leq G$ and $0 \leq r \leq G$.Geometrically this means that the point (q, r) lies in the first quadrant. If we call the integers $a_1, a_2, a_3, \ldots, a_n$ in the the initial state, the "the initial pocket", and if we call the sequence $1^r, 0, 1^r$ the "terminal pocket" and if the terminal pocket were centered over the initial pocket, then either $p = \lceil \frac{G-n+1}{2} \rceil$ or that number plus 1. Thus if $p_0 = \lceil \frac{G-n+1}{2} \rceil$, there is a unique integer δ such that

$$q = G - H + \frac{G(G+1)}{2} - (p_0 + \delta)G$$
$$r = H - \frac{G(G+1)}{2} + (p_0 + \delta)G.$$

We can write down a explict formula for p by solving the following problem. If (x_0, y_0) is a point on the line $x + y = b$ where $b > 0$, find an integer k such that $(x_0 - kb, y_0 + kb)$ is on the line but $0 < x_0 - kb \leq b$. If we solve for k, we find that $\frac{x_0}{b} - 1 \leq b < \frac{x_0}{b}$; that is $k = \lfloor \frac{x_0}{b} - 1 \rfloor$, where $\lfloor x \rfloor$ is the smallest integer greater than or equal to x. Thus

$$p = \lfloor \frac{G+1}{2} - \frac{H}{G} \rfloor.$$

It is an easy induction on n, the number of integers in the initial pocket, to show that $p > 0$. This means that no matter how the sequence $a_1, a_2, \ldots, a_n$ is distributed, the final pocket will be spread on both sides of the initial pocket.

A similar computation using the function M enables us to calculate the number N of k-moves necessary to transform $\mathbf{x}_0$ to $\mathbf{x}_1$. Now

$$M(\mathbf{x}_0) = \sum_{j=G+1}^{G+n} j^2 a_{j-G} = \sum_{s=1}^{n} (s+G)^2 a_s = J + 2GH + G^3.$$

Also

$$M(\mathbf{x}_1) = \sum_{j=G-p+1}^{G-p+q} j^2 + \sum_{j=G-p+q+2}^{G-p+q+r+1} j^2.$$

If we let $W = G - p + 1$ we get

$$M(\mathbf{x}_1) = \frac{(G+W)(G+W+1)(2G+2W+1) - W(W-1)(2W-1)}{6} - (W+q)^2.$$

There is an interesting generalization of this problem which we now consider. The "k-moves" that we have considered have taken two blocks from x_k and have added one block to x_{k-1} and x_{k+1} respectively. Suppose that we take $2s$ blocks from x_k while adding s blocks to x_{k-1} and x_{k-1} respectively, where $s \geq 1$. Such unpiling can be thought of as $(s, 2s, s)$ unpiling. Problem $E3267$ considers $(1, 2, 1)$ unpiling. We shall show that there are nice generalizations for the results stated in (a) and (b).

The key to seeing what the proper generalizations are is to notice that an $(s, 2s, s)$ unpiling can be thought of as a $(1, 2, 1)$ unpiling where we bundle s blocks on any pile into a single Block and think of moving single Blocks of s elements rather than s single blocks. The complication comes from the fact that there may be 'left-over' such single blocks, if $s > 1$. A fruitful way to see this unpiling process to to think of the 'left-over' blocks as being at the bottom of the pile and as not eligible for moving.

From now on our k−moves are centered at x_k but are of the form $(s, 2s, s)$. Notice that if f_k is a k−move we have

$$K(f_k(\mathbf{x})) = K(\mathbf{x})$$
$$L(f_k(\mathbf{x})) = L(\mathbf{x})$$
$$M(f_k(\mathbf{x})) = M(\mathbf{x}) + s.$$

The generalization of (a) is easy to come by. It says that any repeated application of k-moves to the sequence $(0^{st}, 2st + r, 0^{st})$ always leads to the sequence (s^t, r, s^t) after exactly $\frac{1}{3}(t + 1)(t + \frac{1}{2})t$ moves.

The generalization of (b) requires a little preparation. Suppose that $a_1, a_2, \ldots, a_n$ are positive integers designating the number of blocks in each of n piles. For each i with $1 \leq i \leq n$, let

$$a_i = st_i + r_i \qquad \text{where} \qquad 0 \leq r_i < s.$$

The idea is to think of the ith pile as consisting of t_1 Blocks each consisting of s blocks and having r_i single blocks at the bottom of the pile. Now perform a $(1, 2, 1)$ unpiling on the piles having $t_1, t_2, \ldots, t_n$ Blocks. Notice that in the ith pile we have r_i single blocks which have remained untouched in the process. From the results that we found for the $(1, 2, 1)$ process, there are two possibilities which may occur. Either they have a single Block of s blocks on top of them or they have no such single Block on top of them. In either case the total number of blocks remaining is less than $2s$ and hence the unpiling process does not continue.

The following statement is the generalization of (b) we are seeking.

(b) For sufficiently large m, the starting sequence $(0^m, a_1, a_2, \ldots, a_n, 0^m)$ leads inexorably to the sequence

$$(0^{m-p}, s^q, 0, s^r, 0^{m+n-p-q-r-1}) + (0^m, r_1, r_2, \ldots, r_n, 0^m)$$

. The formulas for m, p, q, and r are the same as for the (1,2,1) case where we use $t_1, t_2, t_3, \ldots, t_n$ in place of $a_1, a_2, \ldots, a_n$ The formula for the number of k−moves necessary is the same as that obtained from starting with the $0^m, t_1, t_2, \ldots, t_n, 0^m)$ vector and using only $(1, 2, 1)$ k−moves.